ARO 25347.1-MA

(2)

The Survival Probability Function Of A Target Moving Along A Straight Line In A Random Field Of Obscuring Elements

by

S. Zacks and M. Yadin

May 15, 1990

Technical Report No. 1

Prepared Under Contract DAAL03-89-K-0129 with the U.S. Army Research Office

S. Zacks, Principal Investigator

Center For Statistics, Quality Control and Design
State University of New York
University Center at Binghamton
Binghamton, NY 13901

The Survival Probability Function Of A Target Moving Along A Straight Line In A Random Field Of Obscuring Elements

by

S. Zacks and M. Yadin

State University of New York at Binghamton

and

Technion, Israel Institute of Technology

Abstract

A target is moving along a straight line path. Random portions of the path might be invisible to the hunter (in shadow). Shooting trials are conducted only along the visible segments of the path. An algorithm for the numerical determination of the survival probability of the target is developed. This algorithm is based on the distribution of shadow length which is also developed.

Key Words: *Lines of sight*; *visibility probabilities*; *distributions of shadows*; *survival probability.*

Acknowledgement: Research supported by the U.S. Army Research Office, Contract DAAL03-89-K-0129 with the State University of New York at Binghamton.

1. *Introduction.*

Th ~~The present~~ study is focused on the problem of determining the survival probability of a moving target, which is under attack by a hunter. The target (vehicle, tank, etc.) is moving along a straight line path, which is partially obscured from the hunter by randomly distributed objects (trees, clouds, terrain objects, etc.). The target can be destroyed by the hunter only along the visible segments of the path. Visibility contact between the hunter and the target is needed for tau sub 0 ~~τ_0~~ time units for a shooting trial to occur. In any given shooting trial the probability that the target is destroyed is fixed. If the target survives a shooting trial, another identical trial may be attempted if continuous visibility for τ_0 time units is possible. If the target enters an obscured segment of the path, the shooting trials terminate, until visibility contact is reestablished. Under the above assumptions, if the target has to cross a visible segment of length L, its survival probability can be approximated by the negative exponential function $\exp\{-qL\}$, for suitably chosen constant q, $0 < q <$ infinity ~~∞~~. The problem is that the number of visible segments on the moving path, between two specified points $\mathbf{P}_{\ell}$ sub L and $\mathbf{P}_{u}$ sub u, and their lengths are random variables, whose distributions depend on the characteristics of the random field.

This ~~The present~~ study is based on the model of a random Poisson field of obscuring elements. (KR) This model is presented in Section 2. Under the assumptions of this model, it is relatively simple to derive the conditional distribution of the length of a visible segment on the right hand side (r.h.s.) of a point, $\mathbf{P}_x$, on the path, given that the point $\mathbf{P}_x$ is visible. This distribution is given in Section 3.1. On the other hand, it is more complicated to determine the distribution of the length of a segment which is obscured (in shadow). In Section 3.2 we present the methodology for determining the distribution of the length of shadows. This methodology is based on a theory given by Chernoff and Daly (1957). For a given point $\mathbf{P}_x$ on the moving path, Chernoff and Daly (C-D) define the functional $T(x)$, which is the right hand limit of the shadow to the r.h.s. of P_x, cast by obscuring elements in the field which intersect the ray $\mathbf{R}_x$ from the origin $\mathbf{O}$ through $\mathbf{P}_x$. Employing functions

$K_{\pm}(x, y)$ which are defined in Section 2 and derived explicitly for the standard-uniform case in the Appendix, we express the cumulative distribution function (c.d.f.) of $T(x)$ explicitly. The right hand limit of a shadow to the r.h.s. of $\mathbf{P}_x$, is $U(x) = \lim_{n \to \infty} T^n(x)$, where $T^{n+1}(x) = T(T^n(x))$, for $n = 0, 1, \cdots$, $T^0(x) = x$. The relationship between the c.d.f. of $T^{n+1}(x)$ to that of $T^n(x)$, $n = 0, 1, \cdots$ is discussed in Section 3.2. The distribution of $U(x)$ is obtained as a limit of that of $T^n(x)$. From the distribution of $U(x)$ we obtain the conditional distribution of the right end limit of a shadow to the r.h.s. of $\mathbf{P}_x$, given that $\mathbf{P}_x$ is the first point in the shadow.

In Section 4 we employ the results of Section 3 to approximate the survival probability function $S(x, y)$ along the moving path between the points $\mathbf{P}_x$ and $\mathbf{P}_y$, $x < y$. The function $S(x, y)$ is given by the integral equation

$$S(x, y) = A(x, y) + \int_x^y B(x, w) S(w, y) dw \tag{1.1}$$

where $A(x, y)$ and $B(x, y)$ are defined in terms of the distributions of the lengths of visible and non-visible random segments, as shown in Section 4. An algorithm for the discrete approximation of the solution of (1.1) is given in Section 5. Numerical solutions based on this algorithm are provided there too. A Quick Basic program (version 4.5) for computations can be obtained upon request.

In a previous Technical Report [7] we approximated the survival probabilities by deriving lower and upper bounds to the distribution of the number of shooting trials, N, along the path. The present study provides the method of computing the survival probability function $S(x, y)$, which is required for various applications. With the new algorithms for determining distributions of shadows and survival functions we can tackle problems like the Hunter-Escort problem, which will be discussed in another paper.

2. *The Random Field Model And The Determination Of Visibility Probabilities.*

In the present paper we consider a two dimensional physical model. A generalization to a three dimensional model can be done in a similar fashion to that of Yadin and Zacks [5]. The moving path of the target is a straight line $\mathcal{C}$. The Hunter is located at a point $\mathbf{O}$ (the origin), at distance r from $\mathcal{C}$. Let $\mathcal{U}$ and $\mathcal{W}$ be two straight lines parallel to $\mathcal{C}$, located between $\mathbf{O}$ and $\mathcal{C}$, at distances u and w from $\mathbf{O}$, respectively; $0 < u < w < r$. The obscuring objects are modeled by a countable number of disks of random size, which are centered at random points in the strip $\mathcal{S}$, bounded by $\mathcal{U}$ and $\mathcal{W}$. We consider a cartesian coordinate system in which the y-axis is a straight line through $\mathbf{O}$, perpendicular to $\mathcal{C}$, which intersects $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{C}$ at the point $(0,u)$, $(0,w)$ and $(0,r)$, respectively. A point $\mathbf{P}_x$ on $\mathcal{C}$ has coordinates (x,r).

A random disk is represented by the random vector (X,Y,Z), where (X,Y) are the random coordinates of the center of the disk, and Z is its random radius. Without loss of generality, assume that the sample space of (X,Y,Z) is $\mathcal{S}^* = \mathcal{S} \times [a,b]$, where $0 < a < b < \infty$. Let $\mathcal{B}^*$ be the Borel σ-field on $\mathcal{S}^*$. Let $\{(X_i,Y_i,Z_i),\ i = 1,2,\cdots\}$ represent a sequence of countable random disks measurable w.r.t. the same space $(\mathcal{S}^*,\mathcal{B}^*,P)$. It is assumed that the random vectors are independent and identically distributed (i.i.d.), and have a common distribution $H(x,y,z)$. Let $F(z \mid x,y)$ denote the conditional c.d.f. of the radius Z, given the center (X,Y) is at (x,y). Let $h(x,y)$ be the joint p.d.f. of (X,Y), such that $h(x,y) = 0$ for all $(x,y) \notin \mathcal{S}$. We further assume that the probability that a random disk intersects either $\mathbf{O}$ or $\mathcal{C}$ is zero. Let B be any Borel set in $\mathcal{B}^*$. Let $N\{B\}$ designate the number of random disks with coordinates in B.

If $\{B_1,\cdots,B_m\}$ is any finite partition of $\mathcal{S}^*$, $m = 1,2,\cdots$, it is assumed that the random variables $N\{B_i\}$, $i = 1,\cdots,m$ are independent, having Poisson distributions with expected values

$$\mu\{B_i\} = \lambda \iiint_{B_i} dH(x,y,z), \qquad i = 1,\cdots,m, \tag{2.1}$$

$0 < \lambda < \infty$. Such a random field is called a *Poisson field.* The Poisson field is called *standard-uniform* if $dH(x,y,z) = hI_C(x,y)f(z)dxdydz$, where $0 < h < \infty$, C is a subset of $\mathcal{S}$ which represents the field of view of the Hunter, and $I_C(x,y)$ is the indicator function of C. A point $\mathbf{P}_x$ on $\mathcal{C}$ is said to be *visible* from $\mathbf{O}$, if the ray $\mathbf{R}_x$ from $\mathbf{O}$ through $\mathbf{P}_x$ is not intersected by random disks. In a similar manner we can define the notion of simultaneous visibility of several points on $\mathcal{C}$. In our previous papers [2,3,4] we have introduced the functions $K_+(x,t)$ and $K_-(x,t)$ for $0 < t < \infty$; where $\lambda K_\pm(x,t)$ is the expected number of disks centered in $\mathcal{S}$ between the rays $\mathbf{R}_x$ and $\mathbf{R}_{x\pm t}$, which *do not* intersect $\mathbf{R}_x$. Explicit formulae for $K_\pm(x,t)$, for the standard-uniform case, with a uniform distribution of radii on $[a,b]$, $0 < a < b$, is given in the appendix.

Let $[L,U]$ be an interval of the x-coordinates of the point on $\mathcal{C}$ belonging to a segment of interest. Let $L^* < L$ and $U^* > U$ be properly chosen, and C^* the set in $\mathcal{S}$ (trapez) between the rays $\mathbf{R}_{L^*}$ and $\mathbf{R}_{U^*}$. One can verify that the probability that $\mathbf{P}_x$ is visible, for some $L < x < U$, is

$$\psi(x) = \exp\{-[\mu\{C^*\} - \lambda K_-(x, x-L^*) - \lambda K_+(x, U^*-x)]\}. \tag{2.2}$$

For a formula of the simultaneous visibility of n points in $[L,U]$, see Yadin and Zacks [4].

3. *Distributions Of Length Of Visible And Of Shadowed Segments.*

3.1. *Distributions Of The Length Of Visible Segments.*

In the present section we derive a formula for the conditional c.d.f. of the length of a visible segment to the r.h.s. of $\mathbf{P}_x$, given that $\mathbf{P}_x$ is visible.

Let $I(x)$ be an indicator function which assumes the value 1 if $\mathbf{P}_x$ is visible, and the value zero otherwise.

Let $L(x)$ be the length of the visible segment of $\mathcal{C}$ to the r.h.s. of $\mathbf{P}_x$, i.e.,

$$L(x) = \inf\{y : y \geq x, \prod_{x<u\leq y} I(u) = 1\} - x. \tag{3.1}$$

We derive here the formula for

$$V(l \mid x) = P\{L(x) \le l \mid I(x) = 1\}.$$
(3.2)
$$= 1 - P\{L(x) > l \mid I(x) = 1\}.$$

Let C^* be the set of (x, y) points in $\mathcal{S}$, which was defined in the previous section. We derive the formula of $V(l \mid x)$, for $L < x < U$, and $0 \le l \le u - x$.

Let $C_-(x)$ be the set bounded by $\mathcal{U}$, $\mathcal{W}$ and the rays $\mathbf{R}_{L^*}$ and $\mathbf{R}_x$. Let $C(x, l)$ be the set bounded by $\mathcal{U}$, $\mathcal{W}$ and the rays $\mathbf{R}_x$, $\mathbf{R}_{x+l}$; and $C_+(l+x)$ the set bounded by $\mathcal{U}$, $\mathcal{W}$, R_{l+x} and $\mathbf{R}_{U^*}$. Notice that $C^* = C_-(x) \cup C(x,l) \cup C_+(l+x)$. As before, we denote by $\mu\{C\}$ the expected number of disks having centers at the set C, as given by (2.1). Accordingly,

(3.3)
$$\begin{aligned} P\{L(x) > l, I(x) = 1\} &= \exp\{-[\mu\{C_-(x)\} - \lambda K_-(x, x - L^*)] - \mu\{C(x,l)\} \\ &\quad - [\mu\{C_+(l+x)\} - \lambda K_+(l+x, U^* - l - x)]\} \\ &= \exp\{-\mu\{C^*\} + \lambda[K_-(x, x-L^*) + K_+(l+x, U^* - l - x)]\}. \end{aligned}$$

Dividing (3.3) by (2.2) we obtain

(3.4) $$P\{L(x) > l \mid I(x) = 1\} = \exp\{-\lambda[K_+(x, U^* - x) - K_+(l + x, U^* - l - x)]\}.$$

3.2. *The Distribution of Shadow Length.*

We have denoted by $U(x)$ the right hand limit of the shadow on $\mathcal{C}$ to the r.h.s. of $\mathbf{P}_x$. Let $D(u \mid x)$ denote the conditional c.d.f. of $U(x)$, given that the shadow starts at $\mathbf{P}_x$.

Consider the rays $\mathbf{R}_x$ and $\mathbf{R}_y$ for $y > x$. Let $N(x, y)$ denote the number of disks centered in $\mathcal{S}$, which intersect both $\mathbf{R}_x$ and $\mathbf{R}_y$. Define the functional

(3.5) $$T(x) = \sup\{y : N(x, y) \ge 1\}.$$

Furthermore, let $T^{i+1}(x) = T(T^i(x))$, $i = 0, 1, \cdots$ where $T^0(x) = x$. Obviously, $T^{i+1}(x) \ge T^i(x)$, for all $i \ge 0$, and therefore $U(x) = \lim_{i \to \infty} T^i(x)$. $U(x) - x$ is the length of the shadow

to the r.h.s. of $\mathbf{P}_x$. We derive first the c.d.f. of $T(x)$. Clearly, $\{T(x) > t\} = \{N(x,t) \geq 1\}$. Thus,

$$(3.6) \qquad P\{T(x) \leq t\} = P\{N(x,t) = 0\} = \exp\{-\mu(x,t)\},$$

where $\mu(x,t) = E\{N(x,t)\}$. Furthermore,

$$(3.7) \quad \mu(x,t) = \mu\{C^*\} - \lambda K_+(x, U^* - x) - \lambda K_-(t, t - L^*) + \lambda K_+(x, \tilde{t} - x) + \lambda K_-(t, t - \tilde{t},,$$

where $\tilde{t}$ is the coordinate of the bisector between $\mathbf{R}_x$ and $\mathbf{R}_t$. Notice that, since $K_+(x,0) = K_-(x,0) = 0$ for all x,

$$(3.8) \qquad \begin{aligned} \mu(x,x) &= \lim_{t \downarrow x} \mu(x,t) \\ &= \mu\{C^*\} - \lambda K_+(x, u^* - x) - \lambda K_-(x, x - L^*). \end{aligned}$$

Hence,

$$(3.9) \qquad \lim_{t \downarrow x} P\{T(x) \leq t\} = \psi(x),$$

which is the probability that $\mathbf{P}_x$ is visible. Thus, the c.d.f. of $T(x)$, $H(t;x)$ is zero for $t < x$, it has a jump point at x, $H(x;x) = \psi(x)$, and is absolutely continuous at $t > x$. This property is inherited by the c.d.f. of $T^n(x)$, $H_n(t;x)$. We provide now the recursive relationship between $H_n(t;x)$ and $H_{n-1}(t;x)$. Introduce the bivariate distribution

$$G_n(t_1, t_2; x) = P\{T^{n-1}(x) \leq t_1, T^n(x) \leq t_2\}.$$

Since $\{T^n(x) \leq t\} \subset \{T^{n-1}(x) \leq t\}$,

$$(3.10) \qquad \begin{aligned} H_n(t;x) &= P\{T^n(x) \leq t\} \\ &= G_n(t^*, t; x), \qquad \text{all } t^* \geq t. \end{aligned}$$

For $x < z < y < t$,

$$(3.11) \qquad P\{T^n(x) \leq t \mid T^{n-2}(x) = z, T^{n-1}(x) = y\} = \exp\{-[\mu(y,t) - \mu(z,t)]\}.$$

Indeed, given that $\{T^{n-2}(x) = z, T^{n-1}(x) = y\}$, $\{T^n(x) > t\}$ if and only if, there exists at least one disk which intersects $\mathbf{R}_y$ and $\mathbf{R}_t$, but *does not* intersect $\mathbf{R}_z$. Hence,

$$G_n(t_1, t_2; x) = \int_x^{t_1} \int_z^{t_1} \exp\{-[\mu(u, t_2) - \mu(z, t_2)]\} dG_{n-1}(z, u; x). \tag{3.12}$$

These bivariate c.d.f. can be determined recursively, starting with $G_1(t_1, t_2; x) = H(t_2; x)$ for all $t_1 \le t_2$. Moreover,

$$G_2(t_1, t_2; x) = \int_x^{t_1} e^{-\mu(u,t_2)} \left(\int_x^u e^{\mu(z,t_2)} dz \right) dH(u; x). \tag{3.13}$$

Finally, since $H_{n+1}(t; x) \le H_n(t; x)$ for each $t \ge x$ and all $n = 1, 2, \cdots$ the c.d.f. of $U(x)$ is

$$P\{U(x) \le t\} = \lim_{n \to \infty} H_n(t; x). \tag{3.14}$$

Thus, $P\{U(x) \le t\} = 0$ for all $t < x$, and $\lim_{t \downarrow x} P\{U(x) \le t\} = \psi(x)$. The conditional c.d.f. of $U(x)$, given $\{I(x) = 0\}$ is

$$P\{U(x) \le t \mid I(x) = 0\} = \begin{cases} \dfrac{P\{U(x) \le t\} - \psi(x)}{1 - \psi(x)}, & \text{for } t \ge x \\ 0, & \text{for } t < x. \end{cases} \tag{3.15}$$

We are interested, however, in the conditional c.d.f. $D(u \mid x)$, where $\mathbf{P}_x$ is the first point (the left hand limit) of the random segment in shadow.

Simple geometric considerations yield that the length of a random shadow cast by a *single* disk, having left hand limit at $\mathbf{P}_x$, with center on a line parallel to $\mathcal{U}$ at distance h from $\mathbf{O}$, and disk radius Z, is

$$\tilde{U}(x, h, Z) = r \tan\left(2 \sin^{-1}\left(\frac{Z}{\sqrt{h^2 + x_c^2}} \right) + \tan^{-1}\left(\frac{x}{r} \right) \right) - x, \tag{3.16}$$

where (x_c, h) are the coordinates of the center of the disk, with

$$x_c = \frac{x}{r} h + Z \left(1 + \left(\frac{x}{r} \right)^2 \right)^{1/2}. \tag{3.17}$$

Thus, if $a \le Z \le b$ w.p. 1, the minimal length of shadow starting at x is

(3.18)

$$\tilde{u}_m(x) = r\ \tan\left(2\ \sin^{-1}\left(\frac{(a/w)}{\left(1+\left(\frac{x}{r}+\frac{a}{w}\left(1+\left(\frac{x}{r}\right)^2\right)^{1/2}\right)^2\right)^{1/2}}\right)+\tan^{-1}\left(\frac{x}{r}\right)\right)-x.$$

Finally, since a shadow starting at P_x ends at point $U(x) \ge \tilde{u}_m(x)+x$,

$$D(u \mid x) = \frac{P\{U(x) \le u\} - P\{U(x) \le \tilde{u}_m(x)+x\}}{1-P\{U(x) \le \tilde{u}_m(x)+x\}}, \quad \text{for } u \ge x. \tag{3.19}$$

In Section 5 we provide computing algorithms for the numerical determination of the c.d.f.'s $V(l \mid x)$ and $D(u \mid x)$, and illustrate them with a numerical example.

4. *The Survival Probability Function.*

In the present section we establish the integral equation (1.1). Let $\mathbf{P}_x$ and $\mathbf{P}_y$ be a visible point on C and a point to its right, $L^* < x \le y < U^*$. The Hunter starts shooting trials when the target is at $\mathbf{P}_x$. The attack terminates when the target reaches $\mathbf{P}_y$, if it has not been destroyed before. Let $S(x,y)$ designate the survival probability function. We recognize three exclusive and exhaustive events.

(i) The visible segment to the r.h.s. of $\mathbf{P}_x$ terminates at a point to the right of $\mathbf{P}_y$;

(ii) The visible segment on the r.h.s. of $\mathbf{P}_x$ terminates at a point $\mathbf{P}_t$, $t < y$, and the length of the shadow starting at $\mathbf{P}_t$ is greater than $y-t$.

(iii) The visible segment on the r.h.s. of $\mathbf{P}_x$ terminates at a point $\mathbf{P}_t$, $t < y$, and the length of the shadow to the r.h.s. of $\mathbf{P}_t$ is smaller than $y-t$.

As mentioned in Section 1, the conditional survival probability of a target moving on a

visible segment of length L is $\exp\{-q, L\}$, for some $0 < q < \infty$. Accordingly,

$$
\begin{aligned}
S(x,y) &= e^{-q(y-x)}(1 - V(y - x \mid x)) \\
&\quad + \int_x^y e^{-q(t-x)}(1 - D(y \mid t))dV(t - x \mid x) \\
&\quad + \int_x^y e^{-q(t-x)} \left\{ \int_{\tilde{u}_m(t)+t}^y S(z,y)D'(z \mid t)dz \right\} dV(t - x \mid x),
\end{aligned}
\tag{4.1}
$$

where $D'(z \mid t) = \dfrac{\partial}{\partial z} D(z \mid t)$ is the p.d.f. of $D(z \mid t)$. Notice that $D'(z \mid t) = 0$ for all $t \le z \le \tilde{u}_m(t) + t$. Let $z_m(t) = \tilde{u}_m(t) + t$. $z_m(t)$ is the first term on the r.h.s. of (3.18) with $x = t$. Let $t_m(z)$ be the inverse of $z_m(t)$ then, by changing the order of integration we obtain

$$
\begin{aligned}
&\int_x^y e^{-q(t-x)} \left\{ \int_{\tilde{u}_m(t)+t}^y S(z,y)D'(z \mid t)dz \right\} dV(t - x \mid x) \\
&\quad = \int_x^y S(z,y) \left\{ \int_0^{t_m(z)-x} e^{-qt} D'(z \mid t + x)dV(t \mid x) \right\} dz.
\end{aligned}
\tag{4.2}
$$

Accordingly, define

$$
A(x,y) = e^{-q(y-x)}(1 - V(y - x \mid x)) + \int_0^{y-x} e^{-qt}(1 - D(y \mid t + x))dV(t \mid x), \tag{4.3}
$$

and

$$
B(x,z) = \int_0^{t_m(z)-x)} e^{-qt} D'(z \mid t + x)dV(t \mid x). \tag{4.4}
$$

Thus, the integral equation (4.1) can be written as in (1.1).

5. *Algorithms For Discrete Approximations And Numerical Examples.*

In the present section we consider discrete approximations to the functions $H_n(t; x)$, $G_n(t_1, t_2; x)$, $n \ge 2$ and $S(x,y)$.

For a given integer, N, partition the interval (x,y) to N subintervals. Accordingly, let $\delta = (y - x)/N$, $t_0 = x$ and $t_j = t_0 + j\delta$, $j = 0, 1, \cdots, N$.

For $i = 0, \cdots, N$, let

$$\hat{H}_1(i) = H(t_i; t_0) = \exp\{-\mu(t_0, t_i)\}. \tag{5.1}$$

We compute next the function $\hat{G}_2(i,j)$, $i,j = 0, \cdots, N$; which is a discrete approximation to (3.13). For $i = 0, \cdots, N$ and $j = i, \cdots, N$,

$$\hat{G}_2(i,j) = \sum_{k=0}^{i} \exp\{-(\mu(t_k, t_j) - \mu(t_0, t_j))\} \cdot [\hat{H}_1(k) - \hat{H}_1(k-1)], \tag{5.2}$$

where $\hat{H}_1(-1) = 0$. Notice that $\hat{G}_2(0,j) = \hat{H}_1(0)$ for all $j = 0, 1, \cdots, N$; and for $i \geq 1$, $j \geq i$

$$\hat{G}_2(i,j) = \hat{H}_1(0) + \exp\{\mu(t_0, t_j)\} \cdot \sum_{k=1}^{i} \exp\{-\mu(t_k, t_j)\}[\hat{H}_1(k) - \hat{H}_1(k-1)]. \tag{5.3}$$

Moreover,

$$\hat{G}_2(i,j) = \hat{G}_2(j,j) \quad \text{for all} \quad i > j. \tag{5.4}$$

We compute afterwards recursively, for every $n \geq 3$, $i = 1, \cdots, N$, $j = i, \cdots, N$

$$\begin{aligned} \hat{G}_n(i,j) = \sum_{k=0}^{i} \sum_{l=k}^{i} & \exp\{-(\mu(t_l, t_j) - \mu(t_k, t_j))\}[\hat{G}_{n-1}(k,l) - \hat{G}_{n-1}(k-1,l) \\ & - \hat{G}_{n-1}(k, l-1) + \hat{G}_{n-1}(k-1, l-1)], \end{aligned} \tag{5.5}$$

and for $i > j$

$$\hat{G}_n(i,j) = \hat{G}_n(j,j). \tag{5.6}$$

For $i = 0$, $\hat{G}_n(0,j) = \hat{G}_{n-1}(0,j) = \hat{H}_1(0)$, $j = 0, \cdots, N$. After computing these functions we determine $\hat{H}_n(j) = \hat{G}_n(j,j)$, $j = 0, 1, \cdots, N$. $\hat{H}_n(j)$ is the discrete approximation to the c.d.f. of $T^n(x)$, namely $H_n(t;x)$; i.e., $H_n(t_j; x) \approx \hat{H}_n(j)$.

In Table 5.1 we present numerical results obtained by applying this algorithm to the following special case.

We consider a standard-uniform Poisson field, with uniform distribution for the disk radius on the interval (a, b). In the appendix we present the functions $K_{\pm}(x,t)$, $t \geq 0$, for this case. We compute the numerical example for Table 5.1 with the following geometrical parameters: $r = 100[m]$, $u = 40[m]$, $w = 60[m]$, $a = 1[m]$, $b = 2.5[m]$, $x = 10[m]$, $L^* = -100[m]$, $U^* = 100[m]$. We present in the tables the values of $\hat{H}_n(j)$, $n = 1, 2, 3$, $j = 0, \cdots, 20$, when $\delta = 1[m]$.

Table 5.1. *Values of $\hat{H}_n(j)$ for two values of λ.*

		$\lambda = 0.02$ $[1/m^2]$			$\lambda = 0.2$ $[1/m^2]$	
j	$\hat{H}_1(j)$	$\hat{H}_2(j)$	$\hat{H}_3(j)$	$\hat{H}_1(j)$	$\hat{H}_2(j)$	$\hat{H}_3(j)$
0	0.4914	0.4914	0.4914	0.0000	0.0000	0.0000
1	0.5434	0.5434	0.5434	0.0000	0.0000	0.0000
2	0.6009	0.6008	0.6008	0.0000	0.0000	0.0000
3	0.6645	0.6642	0.6442	0.0003	0.0003	0.0003
4	0.7342	0.7336	0.7336	0.0021	0.0020	0.0020
5	0.8039	0.8030	0.8030	0.0130	0.0119	0.0119
6	0.8664	0.8652	0.8652	0.0576	0.0522	0.0522
7	0.9182	0.9170	0.9170	0.1828	0.1663	0.1663
8	0.9569	0.9559	0.9559	0.4164	0.3843	0.3842
9	0.9811	0.9804	0.9804	0.6842	0.6484	0.6482
10	0.9934	0.9930	0.9930	0.8763	0.8507	0.8505
11	0.9985	0.9984	0.9984	0.9707	0.9590	0.9588
12	0.9999	0.9999	0.9999	0.9981	0.9958	0.9957
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

As seen in Table 5.1, the convergence of $\hat{H}_n(j)$ to the c.d.f. of $U(x)$ is quite rapid. We have therefore approximated the c.d.f. $D(u \mid x)$ by the sequence $\hat{D}(j \mid i) = D(t_j \mid t_i)$, $i = 0, 1, \cdots, N$, $j = i, i+1, \cdots$. The function $A(x, y)$ was computed for the arguments

t_i, t_j, by the approximation

$$
\begin{aligned}
&\hat{A}(N,N) = 1 \\
&\hat{A}(N-1,N) = e^{-q\delta}(1-\hat{V}(1 \mid N-1)) + (1-\tilde{D}(N \mid N-1)) \\
(5.7) \qquad &\hat{A}(N-j,N) = e^{-jq\delta}(1-\hat{V}(j \mid N-j)) \\
&+ \sum_{l=1}^{j} e^{-q(l-\frac{1}{2})\delta}(\hat{V}(l \mid N-j) - \hat{V}(l-1 \mid N-j)) \cdot \\
&\cdot (1-\tilde{D}(N \mid N-j+l)), \quad j = 2, \cdots, N
\end{aligned}
$$

where

$$
(5.8) \qquad \tilde{D}(N \mid N-i) = \frac{1}{2}(\hat{D}(N \mid N-i) + \hat{D}(N \mid N-i+1)),
$$

for all $i = 1, 2, \cdots, N$. Recall that $\hat{D}(N \mid N) = 0 = \hat{D}(N \mid N+1)$.

Similarly, we define

$$
(5.9) \qquad
\begin{aligned}
&\hat{B}(N,N) = 0 \\
&\hat{B}(N-1,N) = \frac{1}{2}e^{-q\delta/2}\hat{V}(1 \mid N-1)\hat{D}(N \mid N-1),
\end{aligned}
$$

and for $j = 2, \cdots, N$, $l = 1, \cdots, j$ we compute

$$
(5.10) \qquad
\begin{aligned}
\hat{B}(N-j, N-j+l) = &\sum_{i=1}^{l} e^{-q\delta(i-\frac{1}{2})}[\hat{V}(i \mid N-j) - \hat{V}(i-1 \mid N-j)] \cdot \\
&\cdot [\hat{D}(N-j+l \mid N-j+i) - \hat{D}(N-j+l-1 \mid N-j+i)],
\end{aligned}
$$

where $\hat{D}(N-j+l-1 \mid N-j+l) = 0$.

Using these sequences, we compute

$$
\hat{S}(N,N) = 1
$$

$$
\hat{S}(N-1,N) = \hat{A}(N-1,N)
$$

and, for $j = 2, \cdots, N$

$$
(5.11) \qquad \hat{S}(N-j,N) = \hat{A}(N-j,N) + \hat{B}(N-j,N) + \sum_{i=N-j+1}^{N-1} \hat{B}(N-j,i)\hat{S}(i,N).
$$

The function $S(x,y)$ is approximated by $\hat{S}(0,N)$. In Table 5.2 we present the values of $\hat{S}(N-j,N)$, for the geometrical parameters of Table 5.1, with $a = 2[m]$, $b = 3.5[m]$ and several values of λ. Also here $\delta = 1[m]$. The value of q is $-ln(0.8)$. This corresponds to the situation in which one shooting trial takes as long as the target travels $1[m]$, and the probability of destroying the target in one trial is 0.8.

Table 5.2. *Survival Probabilities* $\hat{S}(N-j,N)$, *for* $\lambda = 0.02(0.01)0.05$,

$N = 20$; $a = 2$, $b = 3.5$, $r = 100$, $u = 40$, $w = 60$.

$j \backslash \lambda$	0.02	0.03	0.04	0.05
0	1.00000	1.00000	1.00000	1.00000
1	0.81711	0.82447	0.83112	0.83715
2	0.69730	0.72040	0.74052	0.75808
3	0.61877	0.65867	0.69187	0.71965
4	0.56729	0.62203	0.66572	0.70094
5	0.53352	0.60025	0.65163	0.69179
6	0.51135	0.58728	0.64401	0.68729
7	0.49677	0.57954	0.63987	0.68505
8	0.48718	0.57489	0.63759	0.68390
9	0.47460	0.56551	0.63027	0.67817
10	0.45820	0.55111	0.61788	0.66784
11	0.43776	0.53154	0.60001	0.65204
12	0.41376	0.50726	0.57671	0.63029
13	0.38732	0.47959	0.54917	0.60348
14	0.36009	0.45064	0.51979	0.57418
15	0.33382	0.42283	0.49155	0.54589
16	0.30991	0.39803	0.46678	0.52137
17	0.28902	0.37709	0.44652	0.50189
18	0.27122	0.35999	0.43073	0.48741
19	0.25614	0.34607	0.41851	0.47687
20	0.24309	0.33427	0.40849	0.46864

6. *References.*

[1] Chernoff, H. and Daly, J.F. (1957) The Distribution of Shadows, *J. Of Math. and Mechanics* **6**: 567-584.

[2] Yadin, M. and Zacks, S. (1982) Random Coverage of A Circle with Applications to a Shadowing Problem, *J. App. Prob.*, **19**: 562-577.

[3] Yadin, M. and Zacks, S. (1984) The Distribution Of The Random Lighted Portion Of A Curve In A Plane Shadowed By A Poisson Random Field Of Obstacles, *Statistical Signal Processing*, E.J. Wegman and J.G. Smith, Ed. Marcell Dekker, New York.

[4] Yadin, M. and Zacks, S. (1985) The Visibility of Stationary and Moving Targets In The Plane Subject To A Poisson Field of Shadowing Elements, *J. App. Prob.*, **22**: 776-786.

[5] Yadin, M. and Zacks, S. (1988) Visibility Probabilities on Line Segments in Three-Dimensional Spaces Subject to Random Poisson Fields of Obscuring Spheres. *Naval Res. Logist. Quarterly*, **35**: 555-569.

[6] Yadin, M. and Zacks, S. (1986) Discretization of A Semi-Markov Shadowing Process, Tech. Report No. 2, ARO Contract DAAG29-84-K-0191, SUNY at Binghamton, NY.

[7] Zacks, S. and Yadin, M. (1984) Approximating The Probabilities of Detecting And Hitting Targets And The Probability Distribution of The Number of Trials Along The Visible Portions Of Curves In The Plane Subject To A Poisson Shadowing Process, Tech. Report No. 3, ARO Contract DAAG29-83-K-0176, SUNY at Binghamton, NY.

Appendix

The Functions $K_\pm(x,t)$ in the Standard-Uniform Case, With Uniform Distribution of radii on (a,b).

Let $K_+(x,t,z)$ denote the area of the set bounded by the line $\mathcal{L}_z^+$, the ray $\mathbf{R}_{x+t}$, $t \geq 0$, and the lines $\mathcal{U}$ and $\mathcal{W}$; $\mathcal{L}_z^+$ is the line parallel to $\mathbf{R}_x$, on its r.h.s., of distance z from it. This is the set of all disk centers between $\mathbf{R}_x$ and $\mathbf{R}_{x+t}$, of radius $Z = z$, which do not intersect $\mathbf{R}_x$. In order to simplify notation, we assume that $w = r$. In actual computations we substitute xw/r and tw/r for x and t in the formulae given below. Let

$d = (x^2 + w^2)^{1/2}$. Simple geometrical considerations yield:

$$
\text{(A.1)} \qquad \begin{aligned} K_+(x,t,z) &= I\left\{\frac{zd}{t} < u\right\}\left[\frac{w^2-u^2}{2w}t - 2\,\frac{zd}{w+u}\right] \\ &\quad + I\left\{u \le \frac{zd}{t} < w\right\}\left[\frac{1}{2tw}(tw - zd)^2\right] \end{aligned}
$$

where $I\{A\}$ is the indicator set function, which assumes the value 1 if A is true, and the value 0 otherwise.

Notice that $K_+(x,t,z)$ depends on x only via x^2. Symmetry implies that $K_-(-x,t,z) = K_+(x,t,z) = K_+(-x,t,z)$ for all $-\infty < x < \infty$. Hence, $K_+(x,t) = K_-(x,t)$ and we delete the $\pm$ subscript of K. Finally, $K(x,t) = E\{K(x,t,Z)\}$ with respect to the uniform distribution of Z over (a,b). Let $x_1 = tu/d$ and $x_2 = tw/d$. The function $K(x,t)$ assumes the following forms:

(i) If $b < x_1$,

$$
\text{(A.2)} \qquad K(x,t) = \frac{w^2-u^2}{2w}\left(t - \frac{d}{u+w}(a+b)\right).
$$

(ii) If $a < x_1 < b \le x_2$

$$
\text{(A.3)} \qquad \begin{aligned} K(x,t) &= \frac{w^2-u^2}{2w}\left(t\cdot\frac{x_1-a}{b-a} - \frac{d}{u+w}\cdot\frac{1}{b-a}(x_1^2-a^2)\right) \\ &\quad + \frac{1}{2tw}\left(t^2w^2\frac{b-x_1}{b-a} - tw\frac{d}{b-a}(b^2-x_1^2) + \frac{d^2}{3(b-a)}(b^3-x_1^3)\right). \end{aligned}
$$

(iii) If $a < x_1 < x_2 \le b$

$$
\text{(A.4)} \qquad \begin{aligned} K(x,t) &= \frac{w^2-u^2}{2w}\left(t\frac{x_1-a}{b-a} - \frac{d}{u+w}\frac{1}{b-a}(x_1^2-a^2)\right) \\ &\quad + \frac{1}{2tw}\left(t^2w^2\frac{x_2-x_1}{b-a} - tw\frac{d}{b-a}(x_2^2-x_1^2) + \frac{d^2}{3(b-a)}(x_2^3-x_1^3)\right). \end{aligned}
$$

(iv) If $x_1 \le a < b \le x_2$,

$$K(x,t) = \frac{tw}{2} - \frac{d}{2}(a+b) + \frac{d^2(a^2+ab+b^2)}{6tw}. \tag{A.5}$$

(v) If $x_1 \le a < x_2 \le b$

$$K(x,t) = \frac{tw}{2}\frac{x_2 - a}{b-a} - \frac{d}{2(b-a)}(x_2^2 - a^2) + \frac{d^2}{6tw(b-a)}(x_2^3 - a^3). \tag{A.6}$$

(vi) If $x_2 < a$

$$K(x,t) = 0. \tag{A.7}$$

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARO 25347.1-MA	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE SURVIVAL PROBABILITY FUNCTION OF A TARGET MOVING ALONG A STRAIGHT LINE IN A RANDOM FIELD OF OBSCURING ELEMENTS		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S. Zacks and M. Yadin		8. CONTRACT OR GRANT NUMBER(s) DAAL03-89-K-0129
9. PERFORMING ORGANIZATION NAME AND ADDRESS State University of New York Binghamton, NY 13901		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE May 15, 1990 13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

THE VIEW, OPINIONS, AND/OR FINDINGS CONTAINED IN THIS REPORT ARE THOSE OF THE AUTHOR(S) AND SHOULD NOT BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION, POLICY, OR DECISION, UNLESS SO DESIGNATED BY OTHER DOCUMENTATION.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Line of sight; visibility probability; distribution of shadow length; survival probability; Poisson random field.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A target is moving along a straight line path. Random portions of the path might be invisible to the hunter (in shadow). Shooting trials are conducted only along the visible segments of the path. An algorithm for the numerical determination of the survival probability of the target is developed. This algorithm is based on the distribution of shadow length which is also developed.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)